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Manifolds and Topology
Homework 5
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Problem 6.32. (a) Show that if θ is the standard angle function on $\mathbb{R}^2$ measured in the counterclockwise direction, then $d\theta$ is positive on the circle S^1.

To show this is positive on S^1, we need to show that $dr \wedge \pi^* d\theta$ is positive on $\mathbb{R}^n - \{0\}$ where $\pi : \mathbb{R}^2 - \{0\} \to S^1$ is a deformation retraction. Let $\pi(r, \theta) = (1, \theta)$ be the deformation-retraction. Then we have

$$\begin{aligned} \pi^* d\theta &= (1 \circ \pi) d\pi_\theta \\ &= d\theta \end{aligned}$$

So, we are considering now $dr \wedge d\theta$ on $\mathbb{R}^2 - \{0\}$ which is in the orientation class of $\mathbb{R}^2$ so is positive. Thus $d\theta$ is positive on S^1.

(b) Show that if ϕ and θ are the spherical coordinates on $\mathbb{R}^3$ as in Figure 6.7 then $d\phi \wedge d\theta$ is positive on the 2-sphere S^2.

We need to consider $dr \wedge pi^*(d\phi \wedge d\theta)$. Let $\pi : \mathbb{R}^3 - \{0\} \to S^2$ be the deformation retraction given by $\pi(r, \theta, \phi) = (1, \theta, \phi)$. Then we have

$$\begin{aligned} \pi^*(d\phi \wedge d\theta) &= (1 \circ \pi) d\pi_\theta \wedge d\pi_\phi \\ &= d\theta \wedge d\phi \end{aligned}$$

So, we are considering $dr \wedge d\phi \wedge d\theta$ which is in the orientation class of $\mathbb{R}^3$ so is positive, so $d\phi \wedge d\theta$ is also positive.

Problem 6.36. There exist 1-forms ξ_α on U_α such that

$$\frac{1}{2\pi} d\varphi_{\alpha\beta} = \xi_\beta - \xi_\alpha$$

Let $\xi_\alpha = \frac{1}{2\pi} \sum_\gamma \rho_\gamma d\varphi_{\gamma\alpha}$ where $\{\rho_\gamma\}$ is a partition of unity subordinate to $\{U_\gamma\}$. Now, we compute

$$\begin{aligned} \xi_\beta - \xi_\alpha &= \frac{1}{2\pi} \sum_\gamma \rho_\gamma d\varphi_{\gamma\beta} - \frac{1}{2\pi} \sum_\gamma \rho_\gamma d\varphi_{\gamma\alpha} \\ &= \frac{1}{2\pi} \sum_\gamma \rho_\gamma (d\varphi_{\gamma\beta} - d\varphi_{\gamma\alpha}) \end{aligned}$$

Now for each γ, we know that $\varphi_{\gamma\alpha} + \varphi_{\alpha\beta} - \varphi_{\gamma\beta} = 2\pi n_\gamma$ where $n_\gamma \in \mathbb{Z}$. So, we have that $d\varphi_{\gamma\alpha} + d\varphi_{\alpha\beta} - d\varphi_{\gamma\beta} = 0$ since $2\pi n_\gamma$ is a constant function. Thus, we have that $d\varphi_{\alpha\beta} = d\varphi_{\gamma\beta} - d\varphi_{\gamma\alpha}$ and so

$$\begin{aligned}\xi_\beta - \xi_\alpha &= \frac{1}{2\pi}\sum_\gamma \rho_\gamma(d\varphi_{\alpha\beta}) \\ &= \frac{1}{2\pi}(d\varphi_{\alpha\beta})\sum_\gamma \rho_\gamma \\ &= \frac{1}{2\pi} d\varphi_{\alpha\beta}\end{aligned}$$

Problem 6.43. Let $\pi : E \to M$ be an oriented rank 2 bundle. As we saw in the proof of the Thom isomorphism, wedging with the Thom class is an isomorphism $\wedge\Phi : H^*(M) \to H^{*+2}_{cv}(E)$. Therefore every cohomology class on E is the wedge product of Φ with the pullback of a cohomology class on M. Find the class u on M such that

$$\Phi^2 = \Phi \wedge \pi^* u \text{ in } H^*_{cv}(E)$$

Rearranging the given equation, we get that

$$\Phi^2 - \Phi \wedge \pi^* u = \Phi \wedge (\Phi - \pi^* u) = 0$$

By the Thom isomorphism, there is a unique class $\sigma \in H^*(E)$ so that $\Phi \wedge \sigma = 0$. Since $\sigma = 0$ certainly works, we can conlude that $\Phi - \pi^* u = 0$ for a unique class u.
Let e be the Euler class of E. We claim that $u = e$ is the needed cohomology class. Using that $\pi^* e = -d\psi$ and the formula for Φ given in 6.40, we compute

$$\begin{aligned}\Phi - \pi^* e &= d(\rho(r) \wedge \psi) - -d\psi \\ &= d(\rho(r) \wedge \psi + \psi) \\ &= d((\rho(r) + 1) \wedge \psi)\end{aligned}$$

Near 0, $\rho(r) = -1$, so $(\rho(r) + 1) \wedge \psi$ will be defined on all of E despite ψ not being defined near 0. Thus Φ and $\pi^* e$ differ by a closed form, so they represent the same cohomology class. Hence, $u = e$ as claimed.

Problem 6.44. The complex projective space $\mathbb{C}P^n$ is the space of all lines through the origin in $\mathbb{C}^{n+1}$, topologized as the quotient of $\mathbb{C}^{n+1}$ by the equivalence relation

$$z \sim \lambda z \text{ for } z \in \mathbb{C}^{n+1}, \lambda \in \mathbb{C}^\times$$

Let $z_0, \ldots, z_n$ be the complex coordinates on C^{n+1}. These give a set of homogeneous coordinates $[z_0, \ldots, z_n]$ on $\mathbb{C}P^n$ determined up to multiplication by $\lambda \in \mathbb{C}^\times$. Define U_i to be the open subset of $\mathbb{C}P^n$ given by $z_i \neq 0$. $\{U_0, \ldots, U_n\}$ is called the standard open cover of $\mathbb{C}P^n$.

(a) Show that $\mathbb{C}P^n$ is a manifold.

For each i, let $f_i : U_i \to \mathbb{C}^n$ be given by $f_i([z_0, \dots, z_n]) = (\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i})$. To see this is well-defined, notice that

$$\begin{aligned} f_i([\lambda z_0, \dots, \lambda z_n]) &= \left(\frac{\lambda z_0}{\lambda z_i}, \dots, \frac{\lambda z_{i-1}}{\lambda z_i}, \frac{\lambda z_{i+1}}{\lambda z_i}, \dots, \frac{\lambda z_n}{\lambda z_i}\right) \\ &= \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i}\right) \end{aligned}$$

For each i, we also have the inverse map $g_i : \mathbb{C}^n \to U_i$ given by $g_i(w_1, \dots, w_n) = (w_1, \dots, w_{i-1}, 1, w_{i+1}, \dots, w_n)$. Both f_i and g_i are continuous, so we have the needed homeomorphisms.

(b) Find the transition functions of the normal bundle $N_{\mathbb{C}P^1/\mathbb{C}P^2}$ relative to the standard open cover of $\mathbb{C}P^1$.

The normal bundle of $\mathbb{C}P^1$ in $\mathbb{C}P^2$ is given by $\pi : \mathbb{C}P^2 \to \mathbb{C}P^1$ where $\pi([z_0, z_1, z_2]) = [z_0, z_1]$. The standard open cover of $\mathbb{C}P^1$ has only two elements U_0 and U_1, so there is one transition function g_{01}. Each element of the open cover has a corresponding map. For U_0, the fiber is $\{[z_0, z_1, z_2] : z_0 \neq 0\}$ and the map is $\phi_0([z_0, z_1, z_2]) = ([z_0, z_1], \frac{z_2}{z_0})$. Similarly the fiber for U_1 is $\{[z_0, z_1, z_2] : z_1 \neq 0\}$ and the map is $\phi_1([z_0, z_1, z_2]) = ([z_0, z_1], \frac{z_2}{z_1})$. Because in each case we scaled the non-projective output by the nonzero component, the value of z_2 is well-defined and we get an isomorphism to $U_i \times \mathbb{C}$.

Now on $U_0 \cap U_1$, we have the map $\phi_0 \circ \phi_1^{-1}$. It is given by

$$\begin{aligned} (\phi_0 \circ \phi_1^{-1})([a, b], c) &= \phi_0([a, b], bc) \\ &= ([a, b], \tfrac{bc}{a}) \end{aligned}$$

So, the map $g_{01} : U_0 \cap U_1 \to GL_1(\mathbb{C})$ is given by $[z_0, z_1] \mapsto (z \mapsto \frac{z_1}{z_0} z)$.

Problem 6.45. On the complex projective space $\mathbb{C}P^n$ there is a tautological line bundle S, called the universal subbundle; it is the subbundle of the product bundle $\mathbb{C}P^n \times \mathbb{C}^{n+1}$ given by

$$\{(\ell, z) : z \in \ell\}$$

Above each point ℓ in $\mathbb{C}P^n$, the fiber of S is the line represented by ℓ. Find the transition functions of the universal subbundle S of $\mathbb{C}P^1$ relative to the standard open cover and compute its Euler class.

The standard open cover of $\mathbb{C}P^1$ has two elements U_0 and U_1, each with a corresponding map. For U_0, the fiber in the universal subbundle is $\{([1, z], \lambda(1, z)) : z, \lambda \in \mathbb{C}\}$ and the map is $\phi_0([1, z], \lambda(1, z)) = ([1, z], \lambda)$. For U_1, the fiber in the universal subbundle is $\{([z, 1], \lambda(z, 1)) : z, \lambda \in \mathbb{C}\}$ and the map is $\phi_1([z, 1], \lambda(z, 1)) = ([z, 1], \lambda)$.

Now on $U_0 \cap U_1$ we have the map $\phi_0 \circ \phi_1^{-1}$ given by

$$\begin{aligned}
(\phi_0 \circ \phi_1^{-1})([z_0, z_1], \lambda) &= (\phi_0 \circ \phi_1^{-1})([\tfrac{z_0}{z_1}, 1], \lambda) \\
&= \phi_0([\tfrac{z_0}{z_1}, 1], \lambda(\tfrac{z_0}{z_1}, 1)) \\
&= \phi_0([1, \tfrac{z_1}{z_0}], \tfrac{\lambda z_0}{z_1}(1, \tfrac{z_1}{z_0})) \\
&= ([1, \tfrac{z_1}{z_0}], \tfrac{\lambda z_0}{z_1}) \\
&= ([z_0, z_1], \tfrac{\lambda z_0}{z_1})
\end{aligned}$$

So the map $g_{01} : U_0 \cap U_1 \to GL_1(\mathbb{C})$ is given by $[z_0, z_1] \mapsto (z \mapsto \frac{z_0}{z_1} z)$.

For the Euler class, we follow Example 6.44.1. Let $z = \frac{z_1}{z_0}$ be the coordinate of U_0, which we can identify with $\mathbb{C}$. Let $w = \frac{z_0}{z_1} = \frac{1}{z}$ be the coordinate of U_1 which we again identify with $\mathbb{C}$. Then, $g_{01} = \frac{1}{z} = w$ on $U_0 \cap U_1$. Now by 6.38, the Euler class of S is given by

$$\begin{aligned}
e(N) &= \frac{-1}{2\pi i} d\left(\rho_0 d \log g_{01}\right) && \text{on } U_1 \\
&= \frac{-1}{2\pi i} d\left(\rho_0 d \log w\right)
\end{aligned}$$

where ρ_0 is 1 in a neighborhood of the origin and 0 in a neighborhood of infinity in the complex w-plane $U_1 \cong \mathbb{C}$. Let A_r be an annulus centered at the origin whose outer circle C is sufficiently large to contain the support of ρ_0 and whose inner circle B_r has radius r. Orient C counterclockwise and B_r clockwise. Now, we have

$$\int_{\mathbb{C}P^1} e(N) = \frac{-1}{2\pi i} \int_{\mathbb{C}} d\rho_0 d \log w$$

To compute the right integral, consider

$$\begin{aligned}
\int_{\mathbb{C}} d(\rho_0 dw/w) &= \lim_{r \to 0} \int_{A_r} d(\rho_0 dw/w) \\
&= \lim_{r \to 0} \int_C \rho_0 dw/w + \int_{B_r} dw/w && \text{by Stokes' theorem} \\
&= \lim_{r \to 0} \int_{B_r} dw/w \\
&= -2\pi i
\end{aligned}$$

In the third equality, we used that ρ_0 is supported inside of C, so is 0 on C. In the fourth equality, we get a minus sign because B_r is oriented clockwise. So, we have that

$$\int_{\mathbb{C}P^1} e(N) = \frac{-1}{2\pi i}(-2\pi i) = 1$$

So the Euler class is a form with total integral over $\mathbb{C}$ equal to 1.

Problem 6.46. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ and i the antipodal map on S^n:

$$i : (x_1, \dots, x_{n+1}) \to (-x_1, \dots, -x_{n+1})$$

The real projective space $\mathbb{R}P^n$ is the quotient of S^n by the equivalence relation $x \sim i(x)$.

(a) An invariant form on S^n is a form ω such that $i^*\omega = \omega$. The vector space of invariant forms on S^n, denoted $\Omega^*(S^n)^I$ is a differential complex, and so the invariant cohomology $H^*(S^n)^I$ of S^n is defined. Show that $H^*(\mathbb{R}P^n) \cong H^*(S^n)^I$.

Let $\pi : S^n \to \mathbb{R}P^n$ be the quotient by $x \sim i(x)$. This gives an isomorphism on the level of forms. Suppose that ω is an invariant form on S^n. Then $i^*\omega = \omega$ and $\pi(\omega)$ will be well-defined on the quotient of S^n by the antipodal map i. On the other hand, if τ is a form on $\mathbb{R}P^n$ in coordinates $x_1, \dots, x_n$, then $\frac{1}{2}(\tau + i^*\tau)$ is a form on S^n in (non-projective) coordinates $x_1, \dots, x_n$ and is invariant since

$$\tfrac{1}{2}i^*(\tau + i^*\tau) = \tfrac{1}{2}(i^*\tau + \tau)$$

Now when we send this form to $\mathbb{R}P^n$ by identifying $x \sim i(x)$, we will get the form $\frac{1}{2}(\tau + \tau) = \tau$ on $\mathbb{R}P^n$ back.

To see this is an isomorphism in cohomology, we need it to commute with d. Suppose that τ is a form in $\mathbb{R}P^n$. Lifting to S^n, we get the form $\frac{1}{2}(\tau + i^*\tau)$. Applying d, we get $\frac{1}{2}d(\tau + i^*\tau)$. On the other hand, we can compute $d\tau$ in $\mathbb{R}P^n$ and then lift to S^n. This gives $\frac{1}{2}(d\tau + i^*d\tau)$. Since i^* commutes with d, this is $\frac{1}{2}d(\tau + i^*\tau)$. Hence we have an isomorphism in cohomology.

(b) Show that the natural map $H^*(S^n)^I \to H^*(S^n)$ is injective.

Suppose that σ and τ are invariant forms on S^n and that they map to the same element of $\Omega^*(S^n)$. That is, they are the same form when we forget the property that they are invariant. No information about σ and τ is lost when we no longer label them as invariant forms, so they are equal as invariant forms as well. Hence the map is injective at the level of forms.

To see it is injective also on cohomology, suppose that ω is an invariant form on S^n and $[\omega] = 0$, so $\omega = d\tau$ for some (not necessarily invariant) form τ on S^n. Then we have $\omega = i^*\omega = i^*d\tau = di^*\tau$ and so we have that $\omega = \frac{1}{2}(d\tau + di^*\tau) = d\frac{1}{2}(\tau + i^*\tau)$. Now, we compute

$$\begin{aligned} i^*[\tfrac{1}{2}(\tau + i^*\tau)] &= \tfrac{1}{2}(i^*\tau + i^*i^*\tau) \\ &= \tfrac{1}{2}(i^*\tau + \tau) \end{aligned}$$

So $\omega = d\sigma$ where $\sigma = \frac{1}{2}(\tau + i^*\tau)$ is in fact an invariant form on S^n. Thus the map is injective on cohomology.

(c) Give S^n its standard orientation. Show that the antipodal map $i : S^n \to S^n$ is orientation-preserving for n odd and orientation-reversing for n even.

Since S^n inherits its orientation from $\mathbb{R}^{n+1}$, it suffices to consider the effect of i on the orientation of $\mathbb{R}^{n+1}$. So, we need to compute the Jacobian determinant of i. That is, we need to compute $\det(\frac{\partial(-x_i)}{\partial x_i})$. This $(n+1) \times (n+1)$ matrix has -1 on the diagonal and 0 elsewhere, so has value $(-1)^{n+1}$.

When n is odd, this determinant is positive and so orientation preserving. When n is even, this determinant is negative and so orientation reversing.

(d) Show that the de Rham cohomology of $\mathbb{R}P^n$ is

$$H^q(\mathbb{R}P^n) = \begin{cases} \mathbb{R} & q = 0 \\ 0 & 0 < q < n \\ \mathbb{R} & q = n \text{ odd} \\ 0 & q = n \text{ even} \end{cases}$$

We have an isomorphism between $H^q(S^n)^I$ and $H^q(\mathbb{R}P^n)$ and an injection from $H^q(S^n)^I$ to $H^q(S^n)$. Now, we know that

$$H^q(S^n) = \begin{cases} \mathbb{R} & q = 0 \\ 0 & 0 < q < n \\ \mathbb{R} & q = n \end{cases}$$

So since we have an injection from $H^q(\mathbb{R}P^n)$ to $H^q(S^n)$, we can conclude that $H^q(\mathbb{R}P^n) = 0$ for $0 < q < n$. Since $\mathbb{R}P^n$ is connected, we have that $H^0(\mathbb{R}P^n) = \mathbb{R}$. Now when $q = n$, let σ be the generator of $H^n(S^n)$. Since i has Jacobian $(-1)^{n+1}$, we have that $i^*\sigma = (-1)^{n+1}\sigma$. In the case of n odd, σ is an invariant form, and so the injection is in fact an isomorphism and $H^n(\mathbb{R}P^n) = \mathbb{R}$. In the case of n even, σ is not an invariant form and the injection is the 0 map, hence $H^n(\mathbb{R}P^n) = 0$.